ON THE DEFORMATION OF FUCHSIAN GROUPS BY QUASICONFORMAL MAPPINGS WITH PARTIALLY VANISHING BELTRAMI COEFFICIENTS(Dissertation_全文)

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主論文

On the deformation of Fuchsian groups by quasiconformal mappings with partially vanishing Beltrami coefficients

by

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Introduction

Let Γ be a discrete subgroup of the real Möbius group $PSL(2;\mathbb{R})$, and σ be a closed subset of the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ which is invariant under Γ and contains the set $\{0,1,\infty\}$. We denote by D the connected component of $\mathbb{C} - \sigma$ containing the upper half-plane U, that is, $D = U$ or $\mathbb{C} - \sigma$ according as $\sigma = \hat{\mathbb{R}}$ or not. Let $M(\Gamma)$ be the Banach space of those bounded measurable functions μ on D which satisfy

$$\mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z) = \mu(z) \quad \text{for all } \gamma \in \Gamma \text{ and a. e. } z \in D,$$

and furthermore

$$\mu(z) = \bar{\mu}(\bar{z}) \qquad \text{for a. e. } z \in D$$

provided that $\sigma \neq \hat{\mathbb{R}}$. We set

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$$M_1(\Gamma) = \{\ \mu \in M(\Gamma)\ ;\ \|\mu\| = \operatorname*{ess\,sup}_{z \in D} |\mu(z)| < 1\ \}.$$

For μ in $M_1(\Gamma)$. w_μ denotes the unique quasiconformal self-mapping of U which satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$ on U and leaves the points $0, 1, \infty$ fixed. We set

$$M_0(\Gamma,\sigma) = \{\ \mu \in M_1(\Gamma)\ ;\ w_\mu(x) = x \text{ for all } x \in \sigma\ \}.$$

Two elements μ and ν in $M_1(\Gamma)$ are equivalent and written $\mu \sim \nu$ if there exists $\tau \in M_0(\Gamma,\sigma)$ for which $w_\mu \circ w_\tau = w_\nu$. The Teichmüller space $T(\Gamma,\sigma)$ is defined by

$$T(\Gamma,\sigma) = M_1(\Gamma)/M_0(\Gamma,\sigma) = M_1(\Gamma)/\sim ,$$

and the equivalence class of $M_0(\Gamma,\sigma)$ is called the origin of $T(\Gamma,\sigma)$. Note that $T(\Gamma,\hat{\mathbb{R}})$ is the Teichmüller space $T(\Gamma)$ in the usual sense, and $T(\Gamma,\Lambda(\Gamma))$ is the reduced Teichmüller space $T^{\#}(\Gamma)$ if the limit set $\Lambda(\Gamma)$ of Γ contains more than two points. We assume that $T(\Gamma,\sigma)$ is not reduced to a single point. The excluded case occurs only when D_Γ/Γ is (conformally equivalent to) $\mathbb{C} - \{0, 1\}$, where D_Γ is the domain deleted from D the fixed points of all elliptic elements in Γ The Teichmüller space $T(\Gamma,\sigma)$

carries the Teichmüller metric d_T (for the pricise definition see Section 2). The canonical projection: $M_1(\Gamma) \to M_1(\Gamma)/M_0(\Gamma,\sigma) = T(\Gamma,\sigma)$ is open as well as continuous with respect to d_T.

Let V be a subset of D which is invariant under Γ and has positive measure, and let $E = D - V$. When $\sigma \neq \widehat{\mathbb{R}}$, we assume that $V = \{ \bar{z} \, ; \, z \in V \}$, that is, V and E are symmetric with respect to $\mathbb{R}$. We set

$$M(V,\Gamma) = \{ \mu \in M(\Gamma) \; ; \; \mu|E = 0 \},$$

$$M_1(V,\Gamma) = M(V,\Gamma) \cap M_1(\Gamma),$$

and

$$M_0(V,\Gamma,\sigma) = M_0(\Gamma,\sigma) \cap M(V,\Gamma)$$

In this paper we investigate the following two problems;

(A): Under what conditions for V is the set $\{ \mu \in M_1(V,\Gamma) \; ; \; \|\mu\| < \delta \}$, $\delta > 0$, projected to a neighborhood of the origin of $T(\Gamma,\sigma)$?

(B): Under what conditions for V is the origin not an interior point of the image of $M_1(V,\Gamma)$ by the canonical projection : $M_1(\Gamma) \to T(\Gamma,\sigma)$?

When E is a null set, the restriction of the projection to $M_1(V,\Gamma)$ is obviously open and surjective. We remark that the projection does not generally map $M_1(V,\Gamma)$ onto $T(\Gamma,\sigma)$ (Savin [27]), moreover the image of $M_1(V,\Gamma)$ is not necessarily open in $T(\Gamma,\sigma)$ (Oikawa [23]), and that if $\dim T(\Gamma,\sigma) < \infty$, then, for every V with positive measure and every positive δ, $\{ \mu \in M_1(V,\Gamma) \ ; \ \|\mu\| < \delta \}$ is projected to an open neighborhood of the origin of $T(\Gamma,\sigma)$ (Gardiner [7])

Our answers to problem (A) are Theorem 1 and Corollaries 1 and 2 in Section 1, and those to problem (B) are Theorems 2, 3 and 3' in Section 4. We shall prove Theorem 1 and its corollaries in Section 3, after some preliminary facts provided in Section 2. Theorems 2, 3 and 3' will be proved in Section 5. In Section 6 we shall give examples.

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§1. Statements of the answers to problem (A)

Let Ω be a domain in the extended complex plane $\hat{\mathbb{C}}$ which is invariant under a discrete subgroup Γ of $PSL(2;\mathbb{R})$ and satisfies $\infty \notin \Omega$ and $\#(\partial\Omega) \geqq 3$. We denote by $\lambda(z)|dz| = \lambda_\Omega(z)|dz|$ the hyperbolic metric on Ω with constant curvature -4, and by $dA_\Omega(z) = \lambda_\Omega(z)^2 dxdy$ the hyperbolic area element. A measurable automorphic form on Ω of weight -4 for Γ is a measurable function μ on Ω which satisfies

$$\mu(\gamma z)\gamma'(z)^2 = \mu(z) \quad \text{for all} \quad \gamma \in \Gamma \quad \text{and a. e. } z \in \Omega.$$

Such an automorphic form μ is called integrable (resp. bounded) if

$$\|\mu\|_1 = \int_{\Omega/\Gamma} \lambda_\Omega(z)^{-2}|\mu(z)|dA_\Omega(z) < \infty$$

$$\left(\text{resp.} \quad \|\mu\|_\infty = \operatorname*{ess\,sup}_{z \in \Omega} \lambda_\Omega(z)^{-2}|\mu(z)| < \infty \right)$$

We denote by $L^1(\Omega,\Gamma)$ (resp. $L^\infty(\Omega,\Gamma)$) the complex Banach space of all integrable (resp. bounded) automorphic forms on Ω of weight -4 for Γ The closed subspace consisting of all holomorphic elements in $L^p(\Omega,\Gamma)$, $p = 1$ or ∞, is denoted by $A^p(\Omega,\Gamma)$. Furthermore, if Ω is symmetric with respect to $\mathbb{R}$, then we define the real Banach spaces of symmetric elements in $L^p(\Omega,\Gamma)$ and $A^p(\Omega,\Gamma)$ by

$$L^p(\Omega,\Gamma)_{sym} = \{ \mu \in L^p(\Omega,\Gamma) \ ; \ \mu(\bar{z}) = \bar{\mu}(z) \quad \text{for a.e. } z \in \Omega \},$$

and

$$A^p(\Omega,\Gamma)_{sym} = A^p(\Omega,\Gamma) \cap L^p(\Omega,\Gamma)_{sym} ,$$

respectively.

Let D, V and E be the sets as in introduction. For simplicity, we sometimes write L^p (resp. A^p) instead of $L^p(U,\Gamma)$ (resp. $A^p(U,\Gamma)$) when $D = U$, and $L^p(D,\Gamma)_{sym}$ (resp. $A^p(D,\Gamma)_{sym}$) when $D \neq U$. We define

$$L^p(V) = \{ \mu \in L^p \ ; \ \mu|E = 0 \}$$

and

$$A^p|V = \{ \chi(V)\phi \ ; \ \phi \in A^p \},$$

where $\chi(X)$ stands for the characteristic function of a measurable set X. Our assumption that $T(\Gamma,\sigma)$ does not consist of a single point is equivalent to $A^\infty \neq \{0\}$.

Let X and Y be complex (resp. real) Banach spaces, and O be an open set in X. A mapping $f:O \to Y$ is called complex (resp. real) analytic if for each $a \in O$ there exist a positive r and continuous $\mathbb{C}$- (resp. $\mathbb{R}$-) multilinear mappings $A_m:X^m \to Y$, $m \in \mathbb{N}$, such that f has the power series expansion

$$f(x) = f(a) + \sum_{m=1}^{\infty} A_m((x-a)^m)$$

converging absolutely and uniformly on the ball $\{\|x-a\| < r\}$, where $(x-a)^m$ is the element in X^m each entry of which is $x-a$. Standard arguments show that an analytic mapping is of class C^1, that is, it is (Fréchet) differentiable and the derivative is continuous as a mapping of O into the Banach space consisting of all bounded linear operators of X into Y. When X and Y are complex Banach spaces, f is analytic if and only if it is holomorphic, i.e., f is differentiable at each point in O (Mujica [20, Theorem 13.16]). On the other hand, when X and Y are real Banach spaces, f is analytic if and only if there exist an open set O' in $X^{\mathbb{C}}$, the complexification of X, and a holomorphic mapping $F: O' \to Y^{\mathbb{C}}$ such that $O = O' \cap X$ and $f = F|O$.

The following facts are known; the former is implicitly shown in Bers [1] and Earle [3,4]. For a proof of the latter, see Section 2 (also cf. Kra [13]).

Theorem A. In case of $D = U$ (resp. $D \neq U$), there exists an open complex (resp. real) analytic mapping $\Phi_D : M_1(\Gamma) \to A^\infty(L,\Gamma)$ (resp. $A^\infty(D,\Gamma)_{sym}$) such that Φ_D induces a homeomorphism called the Bers embedding of $T(\Gamma,\sigma) = M_1(\Gamma)/M_0(\Gamma,\sigma)$ onto a bounded domain in $A^\infty(L,\Gamma)$ (resp. $A^\infty(D,\Gamma)_{sym}$), where L is the lower half-plane.

From now on we identify $T(\Gamma,\sigma)$ with the bounded domain, namely, the image of $T(\Gamma,\sigma)$ by Φ_D.

Theorem B. There exists a unique function $F = F_{D,\Gamma}$ on $D \times D$ with the following properties:

$$F(\zeta,z) = \bar{F}(z,\zeta),$$

$$F(\eta z,\eta\zeta)\eta'(z)^2\overline{\eta'(\zeta)}^2 = F(z,\zeta) \qquad \text{for every conformal}$$

self-mapping η of D with $\eta\Gamma\eta^{-1} = \Gamma$,

$$F(\cdot,\zeta) \in A^1(D,\Gamma) \cap A^\infty(D,\Gamma) \qquad \text{for each} \quad \zeta \in D,$$

$$\|F(\cdot,\zeta)\|_1 \leqq 3\lambda_D(\zeta)^2,$$

and

$$\phi(z) = \int_{D/\Gamma} \lambda_D(\zeta)^{-4} F(z,\zeta)\phi(\zeta)\,dA_D(\zeta) \ \text{ for } \ \phi \in A^1(D,\Gamma) \cup A^\infty(D,\Gamma).$$

We define a density function ω on D/Γ by

$$\omega(z) = \lambda_D(z)^{-2} \sup_{\zeta \in D} \lambda_D(\zeta)^{-2} |F_{D,\Gamma}(z,\zeta)|.$$

We can now state our answers to problem (A) in introduction; these are generalizations of facts shown and used in Krushkal' [14, p.66], Gardiner [7, 8], Sakan [25, 26], Fehlmann [6] and others.

Theorem 1. Let Γ be a discrete subgroup of $PSL(2;\mathbb{R})$, and σ, D, V; E be the sets as in introduction. Suppose that

$$(1.1) \qquad \int_{E/\Gamma} \max(\omega(z)^2, 1)\, dA_D(z) < \infty .$$

Then there exists $\delta_0 > 0$ such that $\Phi_D(\{ \mu \in M(V,\Gamma) ; \|\mu\| < \delta \})$, $0 < \delta < \delta_0$, is a neighborhood of the origin of $T(\Gamma,\sigma)$ and the restriction of Φ_D to $\{ \chi(V)\lambda_D^{-2}\bar{\psi} ; \psi \in A^\infty, \|\psi\|_\infty < \delta_0 \}$ is an analytic homeomorphism onto an open subset of $T(\Gamma,\sigma)$ Furthermore, to each $\mu \in M_1(V,\Gamma)$ satisfying

$$\iint_{D/\Gamma} \mu\phi\, dxdy = 0 \qquad \text{for all } \phi \in A^1,$$

there exists an analytic curve: $(-\delta_0,\delta_0) \ni t \mapsto \mu(t) \in M_0(V,\Gamma,\sigma)$ such that

$$\mu(0) = 0 \quad \text{and} \quad t\mu = \mu(t) + O(t^2),$$

where the remainder term is uniform with respect to μ, in particular,

$$\left.\frac{d\mu(t)}{dt}\right|_{t=0} = \mu.$$

Corollary 1. If $\mathrm{Area}(E/\Gamma) = \int_{E/\Gamma} dA_D(z) < \infty$ and the condition (1.2): either a Fuchsian model G of Γ contains no hyperbolic elements or

$$\inf \{ |\mathrm{trace}\ g| \ ; \ g \text{ is hyperbolic and in } G \} > 2,$$

is fulfilled, then the conclusion of Theorem 1 holds.

In general, the hypothesis of Corollary 1 is not quasiconformally invariant; namely, there exist Γ, σ, E and a curve $\{ \mu(t) \in M_0(V,\Gamma,\sigma) \ ; \ t \geqq 0 \}$ such that Γ and $w_{\mu(t)}(E)$ satisfy the hypothesis for $t > 0$ but do not for $t = 0$. We shall show this in Section 6.

Corollary 2. If E/Γ is relatively compact in the Riemann surface obtained by adding the punctures of D/Γ to it, then the conclusion of Theorem 1 holds.

The Riemann surface R obtained from D/Γ by adding the punctures may have punctures, e. g., when $D/\Gamma = \mathbb{C} - \mathbb{Z}$, $R = \mathbb{C}$. The hypothesis of Corollary 2 is quasiconformally invariant, hence, in this case, $\Phi_D | M_1(V,\Gamma)$ is an open mapping, in particular, $\Phi_D(M_1(V,\Gamma))$ is open in $T(\Gamma,\sigma)$

§2. Preliminaries

For μ in L^1 and ν in L^∞ the Petersson scalar product is defined by

$$(\mu,\nu) = \int_{D/\Gamma} \lambda_D(z)^{-4}\mu(z)\bar{\nu}(z)\,dA_D(z).$$

Obviously

$$|(\mu,\nu)| \leqq \|\mu\|_1\|\nu\|_\infty \,. \tag{2.1}$$

Note that, when D is symmetric with respect to $\mathbb{R}$, we have

$$(\mu,\nu) = 2\ \mathrm{Re}\int_{U/\Gamma} \lambda_D^{-4}(z)\mu(z)\bar{\nu}(z)\,dA_D(z) \in \mathbb{R}\,.$$

This scalar product establishes isometric isomorphisms; $(L^1)' \cong L^\infty$ and $L^1(V)' \cong L^\infty(V)$, where $(L^1)'$ and $L^1(V)'$ are the dual spaces of L^1 and $L^1(V)$, respectively. We set

$$(A^1)^\perp = \{\ \nu \in L^\infty\ ;\ (\phi,\nu) = 0 \ \text{ for all } \phi \text{ in } A^1\ \}.$$

Let ρ be a universal covering map: $U \to D$ and H the covering group of ρ. For z and ζ in U we set

$$K_U(z,\zeta) = 3/\{\pi(z-\bar{\zeta})^4\}, \tag{2.2}$$

and define a function K_D on $D\times D$ by

$$K_D(\rho z,\rho\zeta)\rho'(z)^2\overline{\rho'(\zeta)}^2 = \sum_{h\in H} K_U(hz,\zeta)h'(z)^2$$

This function is well-defined and independent of the choice of ρ (cf. Kra [13, p.106]), in addition, it has the properties in Theorem B for the case where $\Gamma = \{id.\}$, the trivial group (cf. [13, p.89]) It is not difficult to see that the function $F_{D,\Gamma}$ of Theorem B is given by

$$F_{D,\Gamma}(z,\zeta) = \sum_{\gamma\in\Gamma} K_D(\gamma z,\zeta)\gamma'(z)^2$$

(cf. [13, p.101] and [22])

For μ in $L^1 \cup L^\infty$, define an operator $\beta = \beta_D$ by

(2.3) $$\beta_D[\mu](z) = \int_{D/\Gamma} \lambda_D(\zeta)^{-4} F_{D,\Gamma}(z,\zeta)\mu(\zeta)\,dA_D(\zeta)$$
$$= \int_D \lambda_D(\zeta)^{-4} K_D(z,\zeta)\mu(\zeta)\,dA_D(\zeta), \quad z \in D.$$

This operator is a bounded linear projection of L^1 (resp. L^∞) onto A^1 (resp. A^∞) of norm $\leqq 3$ (cf. [13, p.90, p.101], [22]), and satisfies

(2.4) $$(\beta[\mu].\nu) = (\mu,\beta[\nu]) \quad \text{for } \mu \in L^1, \ \nu \in L^\infty,$$

and

(2.5) $$L^\infty \cap \ker\beta = (A^1)^\perp .$$

To prove Theorem 1 we need an explicit representation of the derivative of Φ_D at $\mu = 0$, which turns out to have a close connection with β_D.

The case $\sigma = \hat{\mathbb{R}}$: For μ in $M_1(\Gamma)$, let w^μ be the unique quasiconformal self-mapping of $\hat{\mathbb{C}}$ which is conformal in the lower half-plane L, satisfies $w_{\bar{z}} = \mu w_z$ in U, and leaves the points $0, i, \infty$ fixed. Let $\{w^\mu\} = (w^\mu)'''/(w^\mu)' - 3\{(w^\mu)''/(w^\mu)'\}^2/2$, the Schwarzian derivative of w^μ in L. The mapping Φ_U: $M_1(\Gamma) \to A^\infty(L,\Gamma)$ is defined by $\Phi_U(\mu) = \{w^\mu\}$. For a proof that Φ_U is a mapping in Theorem A, see Bers [1] or Lehto [17] Here we only check that the (Fréchet) derivative of Φ_U at $\mu = 0$ is given by the formula (2.6) below.

For μ in $M_1(\Gamma)$, ν in $M(\Gamma)$ and t in $\mathbb{C}$ with $|t|$ small, let $f_t = w^{\mu+t\nu} \circ (w^\mu)^{-1}$. Then f_t is a quasiconformal mapping leaving $0, i, \infty$ fixed, whose Beltrami coefficient τ_t vanishes on $w^\mu(L)$ and satisfies $\tau_t = t\tau + O(|t|^2)$, where

$$\tau \circ w^\mu = \frac{\nu}{1 - |\mu|^2} \cdot \frac{(w^\mu)_z^2}{|(w^\mu)_z|^2} .$$

From the variational formula

$$f_t(z) = z - \frac{t}{\pi}\iint_{w^\mu(U)} \frac{z(z-i)\tau(\zeta)}{\zeta(\zeta-i)(\zeta-z)}\,d\xi d\eta + O(|t|^2), \quad \zeta = \xi + i\eta,$$

where the remainder term is uniform for z in each compact subset of $\mathbb{C}$ (cf., for example, Krushkal' [14, p.59]), it follows that

$$\{f_t\}(z) = -\frac{6t}{\pi}\iint_{w^\mu(U)} \frac{\tau(\zeta)}{(\zeta-z)^4}\,d\xi d\eta + O(|t|^2)$$

Since $\{w^{\mu+t\nu}\} = \{f_t \circ w^\mu\} = (\{f_t\}\circ w^\mu)((w^\mu)')^2 + \{w^\mu\}$,

we have

$$\lim_{t\to 0}\frac{1}{t}\left(\Phi_U(\mu+t\nu)(z) - \Phi_U(\mu)(z)\right) = -\frac{6}{\pi}(w^\mu)'(z)^2\iint_{w^\mu(U)} \frac{\tau(\zeta)\,d\xi d\eta}{(\zeta - w^\mu(z))^4}$$

Thus the mapping: $t \mapsto \Phi_U(\mu+t\nu)(z)$ is holomorphic in $\{t \in \mathbb{C}\,;\ \|\mu+t\nu\| < 1\}$ for each fixed z in L. It follows from the Cauchy integral formula that $t \mapsto \Phi_U(\mu+t\nu)$ is an $A^\infty(L,\Gamma)$-valued holomorphic mapping, in other words, $\Phi_U\colon M_1(\Gamma) \to A^\infty(L,\Gamma)$ is Gâteaux differentiable. Since Φ_U is continuous, which is seen from the boundedness of Φ_U by using Schwarz's lemma, Φ_U turns out to be (Fréchet) differentiable in $M_1(\Gamma)$ (Hille [11, Theorem 4.8.1] or Mujica [20, Theorem 8.7]). The derivative at $\mu = 0$, $d\Phi_U(0)$: $M(\Gamma) \to A^\infty(L,\Gamma)$, is given by

$$(2.6) \qquad d\Phi_U(0)[\nu](z) = -\frac{6}{\pi}\iint_U \frac{\nu(\zeta)}{(\zeta - z)^4}\, d\xi d\eta \, .$$

The case $\sigma \neq \hat{\mathbb{R}}$: Let $j(z) = \bar{z}$ and $J(z) = -\bar{z}$, that is, j (resp. J) is the reflection in the real (resp. imaginary) axis. Take a universal covering map $\rho = \rho_U : U \to D$ so that $\rho\circ J = j\circ\rho$ on U. Set $\rho_L = j\circ\rho\circ j$ on L. Since J commutes with j, ρ_L is a universal covering map: $L \to D$ with $\rho_L\circ J = j\circ\rho_L$ Let G be the Fuchsian model of Γ induced by ρ. Note that G is also that of Γ induced by ρ_L For μ in $M(\Gamma)$ and ν in $L^\infty(D,\Gamma)$, we set

$$(2.7) \qquad \rho^*(\mu) = (\mu\circ\rho)\overline{\rho'}/\rho' \quad \text{and} \quad \rho_X^*(\nu) = (\nu\circ\rho_X)(\rho_X')^2 \quad X = U, L.$$

Standard arguments show that $\rho^* : M(\Gamma) \to \{\ \nu \in M(G)\ ;\ \nu\circ J = \bar{\nu}\ \}$ and $\rho_X^* : A^\infty(D,\Gamma) \to A^\infty(X,G)$ are linear isometric isomorphisms, and that

$$\rho_X^*(A^\infty(D,\Gamma)_{sym}) = \{\ \psi \in A^\infty(X,G)\ ;\ \psi\circ J = \bar{\psi}\ \}.$$

The mapping Φ_D is defined by $\Phi_D = (\rho_L^*)^{-1}\circ\Phi_U\circ\rho^*$. Note that $\Phi_U(\{\ \mu \in M_1(G)\ ;\ \mu\circ J = \bar{\mu}\ \}) \subset \{\ \psi \in A^\infty(L,G)\ ;\ \psi\circ J = \bar{\psi}\ \}$. The derivative at $\mu = 0$ is given by

$$d\Phi_D(0)[\mu] = (\rho_L^*)^{-1}\circ d\Phi_U(0)[\rho^*(\mu)].$$

We define a mapping Ψ_D: $\{ \nu \in L^\infty \ ; \ \|\nu\|_\infty < 1 \} \to A^\infty$ by

$$\Psi_D(\nu) = (\overline{\Phi}_D(\lambda_D^{-2}\overline{\nu}))\circ j.$$

When $\sigma = \hat{\mathbb{R}}$, Ψ_D is complex analytic. On the other hand, when $\sigma \neq \hat{\mathbb{R}}$, Ψ_D is only real analytic, however, it is canonically extensible to a complex analytic mapping of the open unit ball in the complex Banach space $L^\infty(D,\Gamma)$ to the complex Banach space $A^\infty(D,\Gamma)$. We have

$$d\Psi_D(0)[\nu] = \overline{d\Phi_D}(0)[\lambda_D^{-2}\overline{\nu}]\circ j \qquad \text{for} \quad \nu \in L^\infty.$$

Hence, in the case $D = U$, by (2.6), (2.2) and (2.3), we see that

$$d\Psi_U(0)[\nu] = -2\beta_U[\nu] \qquad \text{for} \quad \nu \ \text{in} \ L^\infty(U,\Gamma). \tag{2.8}$$

Next, let us consider the case $D \neq U$. It follows from (2.7) that

$$\rho^*(\lambda_D^{-2}\overline{\nu}) = \lambda_U^{-2}\overline{\rho_U^*(\nu)} \qquad \text{for} \quad \nu \ \text{in} \ L^\infty(D,\Gamma).$$

Furthermore, it is not difficult to see that

$$\rho_L^*(\overline{\psi}\circ j) = (\rho_U^*(\psi)\circ j)^- \qquad \text{for} \quad \psi \ \text{in} \ A^\infty(D,\Gamma),$$

$$((\rho_L^*)^{-1}(\overline{\phi}\circ j))^-\circ j = (\rho_U^*)^{-1}(\phi) \qquad \text{for} \quad \phi \ \text{in} \ A^\infty(U,G),$$

and

$$\rho_U^*\circ\beta_D = \beta_U\circ\rho_U^* \qquad \text{on} \quad L^\infty(D,\Gamma)$$

(cf. Kra [13, p.108]). Thus, also in this case, we see

$$(2.9)\qquad \begin{aligned} d\Psi_D(0)[\nu](z) &= ((\rho_L^*)^{-1}(d\Phi_U(0)[\lambda_U^{-2}\overline{\rho_U^*(\nu)}])(\bar{z}))^- \\ &= ((\rho_L^*)^{-1}(\overline{d\Psi_U}(0)[\rho_U^*(\nu)]\circ j)(\bar{z}))^- \\ &= -2\beta_D[\nu](z). \end{aligned}$$

Remark. 1) The Teichmüller distance $d_T([\mu_0],[\nu_0])$ between two points $[\mu_0]$ and $[\nu_0]$ of $T(\Gamma,\sigma) = M_1(\Gamma)/M_0(\Gamma,\sigma)$ (μ_0, $\nu_0 \in M_1(\Gamma)$) is defined by

$$\begin{aligned} d_T([\mu_0],[\nu_0]) &= \frac{1}{2}\inf\{\ \log K(w_\mu \circ (w_\nu)^{-1})\ ;\ \mu \sim \mu_0,\ \nu \sim \nu_0\ \} \\ &= \frac{1}{2}\inf\{\ \log \frac{1 + \|(\mu-\nu)/(1-\bar{\nu}\mu)\|}{1 - \|(\mu-\nu)/(1-\bar{\nu}\mu)\|}\ ;\ \mu \sim \mu_0,\ \nu \sim \nu_0\ \}, \end{aligned}$$

here $K(\cdot)$ denotes the maximal dilatation of a quasiconformal mapping. Since a family of quasiconformal self-mappings of U with uniformly bounded maximal dilatation is normal, there exist μ and ν attaining the infimum of the above definition, and they can be taken so that $\mu = \mu_0$. This shows that the canonical projection : $M_1(\Gamma) \to M_1(\Gamma)/M_0(\Gamma,\sigma)$ is open as well as continuous.

2) Since both Φ_D and the canonical projection are open and continuous, by verifying that $\mu \sim \nu$ if and only if $\{w^\mu\} = \{w^\nu\}$ for μ, $\nu \in M_1(\Gamma)$,

we see that Φ_D induces an embedding of $T(\Gamma,\sigma)$. It is well-known that these two conditions are equivalent when $\sigma = \hat{\mathbb{R}}$ or $\Lambda(\Gamma)$. Suppose that $\Lambda(\Gamma) \subsetneqq \sigma \subsetneqq \hat{\mathbb{R}}$. Then $\mathrm{Cl}(\sigma - \Lambda(\Gamma)) \supset \Lambda(\Gamma)$. Hence it suffices to verify that $w_\mu = w_\nu$ on $\sigma - \Lambda(\Gamma)$ if and only if $w_{\rho^*(\mu)} = w_{\rho^*(\nu)}$ on $\hat{\mathbb{R}}$. This can however be seen by the same argument as used in showing that the Teichmüller space of a Riemann surface is canonically isomorphic to that of a Fuchsian group uniformizing the surface (cf. Lehto [17, Theorem V.1.4]).

§3. Proofs of Theorem 1 and its corollaries

Recalling the definitions of L^∞, $L^\infty(V)$ and other abbreviations, we see that the mapping : $\nu \mapsto \lambda_D^{-2}\bar{\nu}$ (resp. $\psi \mapsto \bar{\psi}\circ j$) is an isometric isomorphism of L^∞ onto $M(\Gamma)$, and of $L^\infty(V)$ onto $M(V,\Gamma)$ (resp. of $A^\infty(L,\Gamma)$ onto $A^\infty(U,\Gamma)$, and of $A^\infty(D,\Gamma)_{sym}$ onto $A^\infty(D,\Gamma)_{sym}$). Hence, by using Ψ_D defined in the preceding section, we can restate Theorem 1 as follows:

Theorem 1'. Under the hypothesis of Theorem 1, there exists a positive δ_0 for which $\Psi_D(\{ \nu \in L^\infty(V) \ ; \ \|\nu\|_\infty < \delta \})$, $0 < \delta < \delta_0$, is an open neighborhood of the origin of A^∞, and the restriction of Ψ_D to $\{ \nu \in A^\infty|V \ ; \ \|\nu\|_\infty < \delta_0 \}$ is an analytic homeomorphism onto an open neighborhood of the origin of A^∞. Furthermore, there exists an analytic mapping τ: $\{ \nu \in L^\infty(V) \cap (A^1)^\perp \ ; \ \|\nu\|_\infty < \delta_0 \} \to L^\infty(V)$ such that $\tau(0) = 0$, $\Psi_D(\tau(\nu)) = 0$ and $d\tau(0) = id$ on $L^\infty(V) \cap (A^1)^\perp$.

We use the following facts to prove Theorem 1'.

Theorem C. Let N be an open set in a Banach space X and $0 \in N$. Let f be a C^1-mapping of N into a Banach space Y with $f(0) = 0$. Suppose that the derivative $df(0) : X \to Y$ is surjective and $\ker df(0)$ splits in X, that is, there is a closed subspace X_1 of X such that X_1 +

$\ker df(0) = X$ and $X_1 \cap \ker df(0) = \{0\}$. Then there exist open sets N', N_1 and N_2 with $0 \in N' \subset N$, $0 \in N_1 \subset X_1$ and $0 \in N_2 \subset \ker df(0)$, and there exist C^1-homeomorphisms h of $N_1 \times N_2$ onto N' with $h(0,0) = 0$, and g of N_1 onto an open subset of Y with $g(0) = 0$ such that the restriction of h to $N_1 \times \{0\}$ is a C^1-homeomorphism onto $N' \cap X_1$,

$$dh(0,0)[(x_1,x_2)] = x_1 + x_2 \quad \text{for} \quad (x_1,x_2) \in X_1 \times \ker df(0)$$

and

$$f \circ h(x_1,x_2) = g(x_1) \qquad \text{for} \quad (x_1,x_2) \in N_1 \times N_2 .$$

In particular, the restriction of f to $N' \cap X_1$ is a C^1-homeomorphism onto $g(N_1)$. If f is analytic, then h and g can be taken so that they are analytic.

For a proof of this theorem, see, for example, Lang [15, Chapter I].

Theorem D. (Bers [2]) The Petersson scalar product induces a linear isomorphism between A^∞ and $(A^1)'$, the dual space of A^1

Lemma 1 Under the hypothesis of Theorem 1, the mapping $\beta_D : L^\infty(V) \rightarrow$

A^{∞} is surjective, and $A^{\infty}|V$ is a closed subspace of $L^{\infty}(V)$ such that

$A^{\infty}|V + (L^{\infty}(V) \cap (A^1)^{\perp}) = L^{\infty}(V)$ and $A^{\infty}|V \cap (A^1)^{\perp} = \{0\}$.

Proof. It has been shown in [22, Theorems 1 and 3] that under the hypothesis of Theorem 1 the second conclusion and

(3.1) $$\sup\{ \|\phi\|_1/\|\chi(V)\phi\|_1 \ ; \ \phi \in A^1 \} < \infty$$

hold. Hence it suffices to show that (3.1) yields the surjectivity of β_D above. Let ψ be an arbitrary element in A^{∞}. By (3.1) and (2.1), the linear functional : $A^1|V \ni \chi(V)\phi \mapsto (\phi,\psi)$ is bounded. Thus, by the Hahn-Banach extension theorem and the F Riesz representation theorem, there is $\nu \in L^{\infty}(V)$ such that $(\chi(V)\phi,\nu) = (\phi,\psi)$ for all ϕ in A^1. By using (2.3), Theorem B and (2.4), we see that $(\chi(V)\phi,\nu) = (\phi,\nu) = (\beta_D[\phi],\nu) = (\phi,\beta_D[\nu])$ for all ϕ in A^1. Theorem D yields $\psi = \beta_D[\nu]$, therefore, $\beta_D : L^{\infty}(V) \to A^{\infty}$ is surjective, q.e.d.

Proof of Theorem 1'. We apply Theorem C as follows; let $X = L^{\infty}(V)$, N = the open unit ball in $L^{\infty}(V)$, $f = \Psi_D|N$ and $Y = A^{\infty}$. Then we have $df(0) = -2\beta_D|L^{\infty}(V)$ and $\ker df(0) = L^{\infty}(V) \cap (A^1)^{\perp}$ by (2.8), (2.9) and

(2.5) Lemma 1 implies that the hypothesis of Theorem C is satisfied. It is easily seen that the conclusion of Theorem C yields that of Theorem 1'. We may take $h(0,\nu)$ as $\tau(\nu)$, q.e.d.

The function ω is bounded under the condition (1.2) (cf. [22, Proposition 2]) Hence Corollary 1 immediately follows from Theorem 1.

Proof of Corollary 2. Let π be the natural projection : $D \to D/\Gamma$ It is shown in the proof of Lehner [16] that there are mutually disjoint punctured disks Δ'_n with finite area in D/Γ and a constant C such that the Riemann surface $D/\Gamma - \bigcup_n (\Delta'_n \cup \partial\Delta'_n)$ has no punctures and

$$\|\chi(N)\phi\|_\infty \leqq C\|\phi\|_1 \qquad \text{for all } \phi \text{ in } A^1(D,\Gamma),$$

where $N = \pi^{-1}(\bigcup_n \Delta'_n)$. Hence, by Theorem B, for all $(z,\zeta) \in N \times D$ we have

$$\begin{aligned}\lambda_D(z)^{-2}\lambda_D(\zeta)^{-2}|F_{D,\Gamma}(z,\zeta)| &\leqq \|\chi(N)\lambda_D(\zeta)^{-2}F_{D,\Gamma}(\cdot,\zeta)\|_\infty \\ &\leqq C\|\lambda_D(\zeta)^{-2}F_{D,\Gamma}(\cdot,\zeta)\|_1 \leqq 3C,\end{aligned}$$

or

$$\omega(z) \leqq 3C \qquad \text{for } z \in N.$$

If E/Γ is relatively compact in the Riemann surface obtained by adding the punctures of D/Γ to it, then so is $E/\Gamma - \bigcup_n \Delta'_n$ in D/Γ and $E/\Gamma \cap \Delta'_n \neq \emptyset$ only for finitely many n. In particular, Area(E/Γ) is finite. Furthermore, since ω is locally bounded in D ([22,Proposition 1]), ω is bounded on $E - N$. Consequently ω is bounded on E. The condition (1.1) in Theorem 1, therefore, holds. This completes the proof.

§4. Statements of the answers to problem (B)

In the following sections we study problem (B). If $\dim T(\Gamma,\sigma) < \infty$, then , as stated in introduction and seen in the sections 2 and 3, $\Phi_D(M_1(V,\Gamma))$ is open in $T(\Gamma,\sigma)$ for every V with positive measure, hence we deal only with the case where $\dim T(\Gamma,\sigma) = \infty$, i.e., $\dim A^1 = \infty$.

Let $\kappa \in M(\Gamma)$. A sequence $\{\phi_n\}_{n=1}^{\infty}$ in A^1 is called a Hamilton sequence for κ if

$$\|\phi_n\|_1 = 1 \quad \text{for all} \quad n \quad \text{and} \quad \lim_{n\to\infty} \iint_{D/\Gamma} \kappa\phi_n \, dxdy = \|\kappa\|.$$

Such a sequence is said to be degenerate if it converges to zero locally uniformly in D. A Beltrami coefficient κ in $M_1(\Gamma)$ (or a quasiconformal mapping w_κ) is said to be extremal if $\|\nu\| \geqq \|\kappa\|$ for all ν in $M_1(\Gamma)$ with $\nu \sim \kappa$, that is, w_κ has the smallest maximal dilatation in its equivalence class. Hamilton, Reich, Strebel and others have shown that κ is extremal if and only if it has a Hamilton sequence.

Remark. 1) If $\dim A^1 < \infty$, then no Hamilton sequences are degenerate.

2) Let $\{\phi_n\}_{n=1}^{\infty}$ be a Hamilton sequence for an extremal κ, and $\lim_{n\to\infty} \phi_n = \phi$.

If $\|\phi\|_1 > 0$, then $\kappa = \|\kappa\|\bar{\phi}/|\phi|$ Moreover, if $0 < \|\phi\|_1 < 1$, then κ has a degenerate Hamilton sequence $\{(\phi_n - \phi)/\|\phi_n - \phi\|_1\}_{n=1}^{\infty}$ (Harrington-Ortel [10]).

3) Suppose that Γ_1 be a normal subgroup of Γ such that the quotient group Γ/Γ_1 is finitely generated and abelian. If $\kappa \in M_1(\Gamma)$ is extremal with respect to $M_0(\Gamma,\sigma)$, then κ, as an element in $M_1(\Gamma_1)$, is also extremal with respect to $M_0(\Gamma_1,\sigma)$ ([21]). In particular, there exists an extremal Beltrami coefficient for which all Hamilton sequences are degenerate.

Let (Δ, d_Δ) be the unit disk Δ equipped with the hyperbolic distance d_Δ, $I = \Delta \cap \mathbb{R}$ and $d_I = d_\Delta|I$. For κ in $M(\Gamma)$ with $\|\kappa\| = 1$, let $\Delta(\kappa) = \{ \zeta\kappa \; ; \; \zeta \in \Delta \}$ and $I(\kappa) = \{ t\kappa \; ; \; t \in I \}$ The following theorem is one of our answers to problem (B).

Theorem 2. Let Γ, D and V be as in introduction. Suppose that

$$\int_{V/\Gamma} \omega(z)\,dA_D(z) < \infty . \tag{4.1}$$

Then for every κ in $M(\Gamma)$, $\|\kappa\| = 1$, with a degenerate Hamilton sequence we have

$$\Phi_D(M_1(V,\Gamma)) \cap \Phi_D(\Delta(\kappa)) = \{0\} \qquad \text{when} \quad D = U,$$

and

$$\Phi_D(M_1(V,\Gamma)) \cap \Phi_D(I(\kappa)) = \{0\} \qquad \text{when} \quad D \neq U.$$

The other answer of ours is Theorem 3 below. For simplicity we restrict ourselves to the case where Γ contains no elliptic elements. To make statements clear, we set $R = U/\Gamma$, $R^* = D/\Gamma$, and simply denote by V both subsets $(V \cap U)/\Gamma$ and V/Γ of R and R^*. respectively. Similar abbreviation is also used for E. We, in addition, define

$$M(R) = \{ \text{ (-1,1)-differentials } \nu \text{ on } R \text{ ; } \|\nu\| < \infty \}$$

and, when $D \neq U$,

$$M(R^*) = \{ \text{ (-1,1)-differentials } \nu \text{ on } R^* \text{ ; } \|\nu\| < \infty, \ \nu \circ J = \bar{\nu} \},$$

where $\|\nu\| = \text{ess sup}|\nu|$ and J is the anti-conformal involution of $R^* = D/\Gamma$ induced by that of $D : z \to \bar{z}$. We identify these two spaces with $M(\Gamma)$, and use the same letters to represent elements in them. The notations $M_1(R)$. $M_1(V,R)$ etc. are self-explanatory.

Let $\Delta_r = \{ |z| < r \}$, $\Delta'_r = \{ 0 < |z| < r \}$, $\Delta = \Delta_1$ and $\Delta' = \Delta'_1$.

Let Ω be a Riemann surface satisfying the following condition:

(4.2) there exist an analytic mapping p of Ω into Δ' and a sequence $\{a_n\}_{n=1}^{\infty} \subset \Delta'$ with $\lim_n a_n = 0$ such that

i) (Ω, p) is a covering surface of Δ',

ii) every point in $p^{-1}(a_n)$ is a branch point for each n,

iii) $(\Omega - p^{-1}(\{a_n ; n \in \mathbb{N}\}), p)$ is a regular (i.e., smooth and complete) covering surface of $\Delta' - \{a_n ; n \in \mathbb{N}\}$, and

iv) the number of sheets of the covering is finite.

Theorem 3. (a) Suppose that R contains Ω above, and the relative boundary $\partial\Omega$ in R consists of finitely many Jordan curves none of which are homotopic to zero in R. Let κ be the canonical extension to R of the lift to Ω of the Beltrami differential $z/\bar{z}$ on Δ', that is, $\kappa \in M(R)$ is given by

$$\kappa | R\text{-}\Omega = 0 \quad \text{and} \quad \kappa(w)\frac{d\bar{w}}{dw} = \frac{z}{\bar{z}} \cdot \frac{d\bar{z}}{dz} \quad \text{for} \quad z = p(w), \quad w \in \Omega.$$

If a measurable subset V of R satisfies

(4.3) $$\iint_{p(\Omega\cap V)-\Delta_r} \frac{dxdy}{|z|^2} = o(\log\frac{1}{r}) \quad \text{as} \quad r \to 0,$$

then we have

$$\Phi_D(M_1(V,R)) \cap \Phi_D(\Delta(\kappa)) = \{0\}$$

and the mapping : $\zeta \mapsto \Phi_D(\zeta\kappa)$ is an isometry of (Δ, d_Δ) into $(T(\Gamma,\sigma), d_T)$

(b) Let V be a measurable subset of R^* which is invariant under the anti-conformal involution J of R^*. Suppose that R^* contains Ω, and $\partial\Omega$ consists of finitely many Jordan curves none of which are homotopic to zero in R^*. If (4.3) and the following condition:

(4.4) $\Delta' - \{a_n ; n \in \mathbb{N}\}$ is symmetric with respect to $\mathbb{R}$, $J(\Omega) = \Omega$ and $J|\Omega$ is projected by p to the involution of $\Delta' - \{a_n ; n \in \mathbb{N}\} : z \mapsto \bar{z}$,

hold, then we have

$$\Phi_D(M_1(V,R^*)) \cap \Phi_D(I(\kappa)) = \{0\}$$

and the mapping : $t \mapsto \Phi_D(t\kappa)$ is an isometry of (I, d_I) into $(T(\Gamma,\sigma), d_T)$.

The condition (4.3) does not necessarily imply the hypothesis (4.1) of Theorem 2. We shall show this in Section 6.

Theorem 3 is generalized as follows:

Theorem 3'. Let Ω_j, Ω'_k ($j = 1,\dots, N$, $k = 1,\dots, N'$, $1 \leqq N + N' \leqq \infty$, and $N' = 0$ if $R^* = R$) be mutually disjoint $N + N'$ subdomains of R^* satisfying (4.2). Suppose that, for each j and k, $\Omega_j \subset R$, Ω'_k satisfies (4.4), and $\partial\Omega_j$, $\partial\Omega'_k$ consist of finitely many Jordan curves none of which are homotopic to zero in R^*. Let V be a measurable subset of R^* satisfying (4.3) for each Ω_j, Ω'_k, and furthermore $J(V) = V$ if $R^* \neq R$. Then we have

$$\text{(4.5)} \qquad \Phi_D(M_1(V,R)) \cap \Phi_D(\{ \sum_{j=1}^{N} \zeta_j\kappa_j \ ; \ \zeta_j \in \Delta \}) = \{0\} \quad \text{when} \quad R = R^*.$$

and

$$\Phi_D(M_1(V,R^*)) \cap \Phi_D(\{ \sum_{j=1}^{N} (\zeta_j\kappa_j + \bar{\zeta}_j\bar{\kappa}_j \circ J) + \sum_{k=1}^{N'} t_k\kappa'_k \ ; \ \zeta_j \in \Delta,\ t_k \in I\})$$
$$= \{0\} \qquad \text{when} \quad R \neq R^*,$$

where κ_j and κ'_k are the Beltrami differentials as in Theorem 3 for Ω_j and Ω'_k, respectively. Moreover, $\Delta^N \ni (\zeta_j) \mapsto \Phi_D(\sum_1^N \zeta_j\kappa_j) \in T(\Gamma,\sigma)$ and $\Delta^N \times I^{N'} \ni ((\zeta_j),(t_k)) \mapsto \Phi_D(\sum_1^N(\zeta_j\kappa_j + \bar{\zeta}_j\bar{\kappa}_j \circ J) + \sum_1^{N'} t_k\kappa'_k) \in T(\Gamma,\sigma)$ are isometries, where the distance between (ζ_j) and (ζ'_j) in Δ^N (resp. $((\zeta_j),(t_k))$ and $((\zeta'_j),(t'_k))$ in $\Delta^N \times I^{N'}$) is defined by $\sup_j d_\Delta(\zeta_j,\zeta'_j)$ (resp. $\max\{\sup_j d_\Delta(\zeta_j,\zeta'_j),\ \sup_k d_I(t_k,t'_k)\}$).

§5. Proofs of Theorems 2, 3 and 3'

Let S be a Riemann surface whose universal covering surface is U, and K be the covering transformation group. We denote by $b(S)$ the border $(\hat{\mathbb{R}} - \Lambda(K))/K$ of $S = U/K$; $b(S)$ may be empty. Every quasiconformal mapping of S onto another Riemann surface S' extends to a homeomorphism between the bordered Riemann surfaces $S \cup b(S)$ and $S' \cup b(S')$.

Let $X_0 \subset X \subset S \cup b(S)$ and $Y \subset S' \cup b(S')$ Two continuous mappings f and g of X into Y are said to be homotopic relative to X_0 if there is a homotopy $h : X \times [0,1] \to Y$ from f to g such that

$$h(x,t) = f(x) = g(x) \qquad \text{for all } (x,t) \in X_0 \times [0,1]$$

We then write $f \simeq g : X \to Y$ rel X_0 or $f \simeq g$ rel X_0 for short, and if $X_0 = \emptyset$ then we often omit " rel X_0 ".

Let Γ, σ be as in introduction. For $\mu \in M_1(\Gamma)$ let f_μ be the quasiconformal mapping of the Riemann surface D_Γ/Γ induced by the quasiconformal mapping w_μ of D, where w_μ is regarded to be extended to L by symmetry when D is symmetric with respect to $\mathbb{R}$. Then, for μ and ν in $M_1(\Gamma)$,

μ and ν are equivalent with respect to $M_0(\Gamma,\sigma)$ if and only if f_μ and f_ν are homotopic relative to $(\sigma - \Lambda(\Gamma))/\Gamma$ (cf. Lehto [17, p.180] and Marden [19])

Theorem E. (A short form of the main inequality of Reich and Strebel)
Let f and g be quasiconformal mappings of a Riemann surface S which are homotopic relative to a closed subset δ of $b(S)$. Let κ and ν_1 be the Beltrami coefficients of f and g^{-1}, respectively. Then for every integrable holomorphic quadratic differential ϕ on S which is real on $b(S) - \delta$, we have

$$\iint_S |\phi|\,dxdy \leqq \iint_S |\phi| \frac{|1-\kappa\phi/|\phi||^2}{1-|\kappa|^2} \cdot \frac{1+|\nu_1\circ g|}{1-|\nu_1\circ g|}\,dxdy$$

Note that an integrable holomorphic quadratic differential on S is real on $b(S) - \delta$ if and only if it can be lifted to (the restriction to U of) an element in $A^1(\hat{\mathbb{C}} - \tilde{\delta},K)_{sym}$.

For a proof of the above theorem, see Strebel [28].

Lemma 2. ([22, Lemma 3]) Under the hypothesis of Theorem 2, for a sequence $\{\phi_n\}_{n=1}^\infty$ in A^1 with $\|\phi_n\|_1 = 1$ and $\lim_n \phi_n = 0$, we have

$\lim_n \|\chi(V)\phi_n\|_1 = 0$

As a by-product of Lemma 2 we immediately obtain the following, which is well-known for the case where V/Γ is relatively compact in D/Γ

Proposition 1 Let Γ, D, V be as in Theorem 2. If $\kappa \in M_1(\Gamma)$ has a degenerate Hamilton sequence, then every $\nu \in M_1(\Gamma)$ satisfying

$$\nu|D-V = \kappa|D-V \quad \text{and} \quad \|\nu|V\| \leqq \|\kappa\|$$

is extremal.

Proof of Theorem 2. Suppose that there exist $\zeta\kappa \in \Delta(\kappa)$ and $\nu \in M_1(V,\Gamma)$ such that $\Phi_D(\zeta\kappa) = \Phi_D(\nu) \neq 0$. Let $\{\phi_n\}$ be a degenerate Hamilton sequence for κ, then so is $\{\bar{\zeta}\phi_n/|\zeta|\}$ for $\zeta\kappa$. Applying Theorem E to $w_{\zeta\kappa}$, w_ν and $\bar{\zeta}\phi_n/|\zeta|$, we have

$$1 \leqq \frac{1}{1-|\zeta|^2} \iint_{E/\Gamma} |\phi_n| \left|1 - \frac{|\zeta|\kappa\phi_n}{|\phi_n|}\right|^2 dxdy + \frac{1+|\zeta|}{1-|\zeta|}\,\frac{1+\|\nu_1\|}{1-\|\nu_1\|}\|\chi(V)\phi_n\|_1 ,$$

We note that $\nu_1 \circ w_\nu = 0$ on E. Lemma 2 implies

$$1 - |\zeta|^2 \leqq \iint_{D/\Gamma} |\phi_n| \left|1 - |\zeta|\kappa\phi_n/|\phi_n|\right|^2 dxdy + o(1)$$

$$\leqq 1 + |\zeta|^2 - 2|\zeta| \operatorname{Re} \iint_{D/\Gamma} \kappa\phi_n dxdy + o(1)$$

$$= (1 - |\zeta|)^2 + o(1),$$

a contradiction to $\zeta \neq 0$, q.e.d.

Lemma 3. Let $\phi(z,a) = 1/\{z(z-a)\}$ $(0 < |a| \leqq 1)$, and let V, (Ω, p) be as in Theorem 3. Then we have

$$(5.1) \qquad \iint_\Delta \frac{z}{\bar{z}} \phi(z,a)\,dxdy = 2\pi|\log|a||,$$

$$(5.2) \qquad \|\phi(\cdot,a)\|_1 = 2\pi|\log|a|| + O(1) \quad \text{as} \quad a \to 0,$$

and

$$(5.3) \qquad \iint_{p(V \cap \Omega)} |\phi(z,a)|\,dxdy = o(|\log|a||) \quad \text{as} \quad a \to 0.$$

Proof. The left-hand side of (5.1) is equal to

$$\frac{1}{i}\int_0^1 \frac{dr}{r}\int_{|z|=r} \frac{dz}{z-a} = 2\pi\int_{|a|}^1 \frac{dr}{r}$$

This yields the equality (5.1). Next, since $|\phi(z,a) - z^{-2}| \leqq 2|a||z|^{-3}$ on $A = \{z \,;\, 2|a| \leqq |z| < 1\}$, we have

$$(5.4) \qquad \iint_A \Big||\phi(z,a)| - |z|^{-2}\Big|\,dxdy \leqq 2\pi.$$

The estimate (5.2) follows from

$$\iint_A |z|^{-2}\,dxdy = 2\pi|\log(2|a|)|,$$

and

$$\iint_{\Delta - A} |\phi(z,a)|\,dxdy = \text{const.} \tag{5.5}$$

The last estimate (5.3) follows from (5.4), (5.5) and (4.3) similarly, q.e.d.

Lemma 4. Let S be a Riemann surface, and $b = \bigcup_{j=1}^{n} b_j$ be a union of components of $b(S)$ such that each b_j is a closed curve. Let f and g be continuous mappings of $S \cup b$ into another Riemann surface S' Then $f \simeq g : S \cup b \to S'$ if and only if $f|S \simeq g|S : S \to S'$.

Proof. For each j, let A_j be an annular half-neighborhood of b_j, i.e., A_j is an annular subdomain of S such that one component of $b(A_j)$ is b_j. We can assume that $A_1, .., A_n$ are mutually disjoint. Let $z_j :$ $A_j \cup b_j \to \{ 1/2 < |z| \leqq 2 \}$ be a homeomorphism. We define a continuous mapping $r : S \cup b \to S$ by

$$r(p) = \begin{cases} z_j^{-1}(z_j(p)/|z_j(p)|) & \text{for } p \in \bigcup_{j=1}^{n} z_j^{-1}(\{ 1 \leqq |z| \leqq 2 \}), \\ p & \text{otherwise.} \end{cases}$$

Obviously, $r \simeq id_{S \cup b}$. Hence, if $f|S \simeq g|S$, then $f \simeq (f|S)\circ r \simeq (g|S)\circ r \simeq g$. The converse is trivial, q.e.d.

Let C be an analytic Jordan curve in a Riemann surface S which does not bound a disk nor a one-punctured disk. Take a closed parametric annular neighborhood (N,z) of C such that $N = \{ p \,;\, 1/a \leqq |z(p)| \leqq a \}$ and $C = \{ p \,;\, |z(p)| = 1 \}$. For a non-negative smooth function θ on $(1/a,a)$ with compact support and $\int_{(1/a,a)} \theta(r)dr = 2\pi$, we define a quasiconformal self-mapping τ_C of S by

$$\tau_C | N : z \mapsto z \exp\left(i\int_{1/a}^{|z|} \theta\, dr\right) \quad \text{and} \quad \tau_C | S-N = \mathrm{id}_{S-N}.$$

This mapping τ_C is called a Dehn twist about C. The homotopy class of τ_C does not depend on the assignment of an orientation of C nor the choice of (N,z).

Lemma 5. Let S be a Riemann surface whose universal covering surface is U, and $C_1, \dots, C_n$ be mutually disjoint analytic Jordan curves in S such that no components of $S - \bigcup_{j=1}^{n} C_j$ are disks nor one-punctured disks. Let f and g be quasiconformal mappings of S onto another Riemann surface S'. Suppose that, for each component S_0 of $S - \bigcup_{j=1}^{n} C_j$, $f|S_0 \simeq g|S_0 : S_0 \to S'$ Then $f \simeq g \circ \tau_{C_1}^{m(1)} \circ \cdots \circ \tau_{C_n}^{m(n)} : S \to S'$ for some $(m(1), \dots, m(n)) \in \mathbb{Z}^n$

Proof. Without loss of generality we may assume that $g = id_S$. Let γ be a directed arc in $S - \bigcup_{j=2}^{n} C_j$ intersecting $C = C_1$ with one point, say q, and whose initial and terminal points, say p_1 and p_2 respectively, lie outside the annular neighborhood N of C. Let γ_1 (resp. γ_2) be the subarc of γ with the initial point p_1 (resp. q) and the terminal point q (resp. p_2). Let S_k ($k = 1, 2$) be the component of $S - \bigcup_{j=1}^{n} C_j$ in which p_k lies. We first treat the case where $S_1 \neq S_2$, that is, C is a dividing curve of $S_1 \cup C \cup S_2$. By Lemma 4 there is a homotopy $h_k : (S_k \cup C) \times [0,1] \to S$ from $id_{S_k \cup C}$ to $f|S_k \cup C$. Set $\alpha_k = \{ h_k(p_k,t) \ ; \ 0 \leqq t \leqq 1 \}$ and $\beta_k = \{ h_k(q,t) \ ; \ 0 \leqq t \leqq 1 \}$. Then there is an integer $m = m(1)$ for which $[\alpha_2^{-1} f(\gamma) \alpha_1] = [\gamma_2 \beta_2^{-1} \beta_1 \gamma_1] = [\tau_C^m(\gamma)]$, where square brackets denote an equivalence class with respect to homotopies fixing the initial and terminal points. It is not difficult to see that $[\alpha_1^{-1} f(\alpha) \alpha_1] = [\tau_C^m(\alpha)]$ for such m and every closed curve α in $S_0 = S_1 \cup C \cup S_2$ whose initial and terminal point is p_1, that is to say. (*): $\alpha \mapsto \tau_C^m(\alpha)$ and $\alpha \mapsto \alpha_1^{-1} f(\alpha) \alpha_1$ define the same injective homomorphism of the fundamental group $\pi_1(S_0,p_1)$ of S_0 with base-point p_1 into $\pi_1(S,p_1)$. A slight modification of the above argument shows that (*) is valid also for the case where $S_1 = S_2$, i.e., C is a non-dividing curve of $S_0 = S_1 \cup C$.

Let $\pi : U \to S$ be a universal covering map, and K be the covering transformation group of π. Take a component $\tilde{S}_0$ of $\pi^{-1}(S_0)$, and let $K_0 = \{ \eta \in K ;\ \eta(\tilde{S}_0) = \tilde{S}_0 \}$, the stabilizer of $\tilde{S}_0$. Fix a point $\zeta \in \tilde{S}_0$ over p_1, and let $\tilde{\tau}$ be the lift of τ_C^m such that $\tilde{\tau}(\zeta) = \zeta$. Let ζ' be the terminal point of that lift of α_1 whose initial point is ζ, and $\tilde{f}$ be the lift of f such that $\tilde{f}(\zeta) = \zeta'$. Then one can see that $K_0 \ni \eta \mapsto \tilde{f}\circ\eta\circ\tilde{f}^{-1}$ and $K_0 \ni \eta \mapsto \tilde{\tau}\circ\eta\circ\tilde{\tau}^{-1}$ define the same isomorphism θ of K_0 onto a subgroup K' of K. Hence there is a homotopy $\tilde{h} : U \times [0,1] \to U$ from $\tilde{\tau}$ to $\tilde{f}$ such that $\tilde{h}(\eta z, t) = \theta(\eta)(\tilde{h}(z,t))$ for $\eta \in K_0$, $z \in U$ and $t \in [0,1]$ (cf. Lehto [17, Theorem IV 3.5] or Marden [19]) This homotopy can be projected to a homotopy $h : (U/K_0) \times [0,1] \to U/K'$. Let π' be the canonical projection : $U/K' \to U/K = S$, and consider a continuous mapping : $(\tilde{S}_0/K_0) \times [0,1] \ni (p,t) \mapsto \pi'(h(p,t)) \in S$. This is a homotopy : $S_0 \times [0,1] \to S$ from $\tau_C^m | S_0$ to $f | S_0$. By repeating this argument n times, we obtain the conclusion, q.e.d.

Lemma 6. Let S be a Riemann surface which is different from a disk and a one-punctured disk, and whose border $b(S)$ consists of finitely many closed curves $b_1, \dots, b_n$, and let $A_1, \dots, A_n$ be mutually disjoint annular half-neighborhoods of $b_1, \dots, b_n$. Let f and g be quasiconformal mappings of S onto another Riemann surface S' such that $f \simeq g : S \to S'$ and $f = g$ on $b(S)$. Then there is a quasiconformal mapping g' of S onto S' such that $g' = g$ on $S - \bigcup_{j=1}^{n} A_j$ and $g' \simeq f : S \cup b(S) \to S' \cup b(S')$ rel $b(S)$.

Proof. We may assume $g = id_S$ again. Let p be a point in $S - \bigcup_1^n A_j$, and set $\alpha = \{ h(p,t) \; ; \; 0 \leqq t \leqq 1 \}$, where h is a homotopy from id_S to f (not necessarily fixing the points in $b(S)$). For each j, take a point x_j in b_j, and choose an analytic Jordan curve C_j and an annular neighborhood N_j of C_j so that $N_j \subset A_j$ and C_j is freely homotopic to b_j. Then there is an integer $m(j)$ such that $[\tau_{C_j}^{m(j)}(\gamma)] = [f(\gamma)\alpha]$ for every arc γ connecting p and x_j. This yields that $[\tau_{C_1}^{m(1)} \circ \cdots \circ \tau_{C_n}^{m(n)}(\gamma)] = [f(\gamma)\alpha]$ for all arcs γ connecting p and points in $b(S)$. Furthermore, it is obvious that, for all closed curves γ with initial and terminal point p, $[\alpha^{-1}f(\gamma)\alpha] = [\gamma] = [\tau_{C_1}^{m(1)}\circ\cdots\circ\tau_{C_n}^{m(n)}(\gamma)]$. Hence similar argument as in the proof of Lemma 5 shows that $\tau_{C_1}^{m(1)}\circ\cdots\circ\tau_{C_n}^{m(n)}$ is the required mapping g',

q.e.d.

Lemma 7. Let S be a Riemann surface whose universal covering surface is U, and W be a subdomain of S such that the relative boundary ∂W in S consists of finitely many Jordan curves, none of which are homotopic to zero in S. Suppose that f and g are continuous mappings of another Riemann surface S_0 into W such that $f \simeq g : S_0 \to S$ and the image of the homomorphism $f_* : \pi_1(S_0,p) \to \pi_1(W,f(p))$, derived from f, is not cyclic. Then $f \simeq g : S_0 \to W$.

Proof. Let $\pi : U \to S$ be a universal covering map, and K be the covering transformation group of π. Let W_0 be a component of $\pi^{-1}(W)$, and K_0 be the stabilizer of W_0. Set $\tilde{S} = U/K_0$, $\tilde{W} = W_0/K_0$, and let $\tilde{\pi}$: $\tilde{S} \to S$ be the canonical projection. Then $\tilde{S}$ is a regular covering surface of S, $\tilde{\pi}|\tilde{W}$ is a conformal homeomorphism of $\tilde{W}$ onto W, $\partial\tilde{W}$ consists of finitely many Jordan curves, and each component of $\tilde{S} - (\tilde{W} \cup \partial\tilde{W})$ is an annulas.

Let $h : S_0 \times [0,1] \to S$ be a homotopy from f to g. Since there is a lift $\tilde{f} = (\tilde{\pi}|\tilde{W})^{-1} \circ f : S_0 \to \tilde{W} \subset \tilde{S}$ of $f : S_0 \to W \subset S$, the homotopy h can be lifted to a homotopy $\tilde{h} : S_0 \times [0,1] \to \tilde{S}$ from $\tilde{f}$ to a lift $\tilde{g}$ of g. Suppose that $\tilde{g}(S_0) \not\subset \tilde{W}$. Since $g(S_0) \subset W$, $\tilde{g}(S_0)$ is contained in a component of $\tilde{S} - (\tilde{W} \cup \partial\tilde{W})$. Then $\tilde{g}_*(\pi_1(S_0,p))$ is a cyclic subgroup of $\pi_1(\tilde{S},\tilde{g}(p))$. This and the facts $\tilde{g}_* = \tilde{f}_*$, $\pi_1(\tilde{S}) \cong \pi_1(\tilde{W}) \cong \pi_1(W)$ imply that $f_*(\pi_1(S_0,p))$ is a cyclic subgroup of $\pi_1(W,f(p))$, which contradicts to the hypothesis. Consequently, $\tilde{g}(S_0) \subset \tilde{W}$. Let $r : \tilde{S} \times [0,1] \to \tilde{S}$ be a homotopy from $\mathrm{id}_{\tilde{S}}$ to a continuous mapping $: \tilde{S} \to \tilde{W}$ as in the proof of Lemma 4. For $p \in S_0$, define

$$h'(p,t) = \begin{cases} \tilde{\pi}(r(\tilde{f}(p),3t)), & 0 \leqq t < 1/3, \\ \tilde{\pi}(r(\tilde{h}(p,3t-1),1)), & 1/3 \leqq t \leqq 2/3, \\ \tilde{\pi}(r(\tilde{g}(p),3-3t)), & 2/3 < t \leqq 1. \end{cases}$$

Then this is the required homotopy $: S_0 \times [0,1] \to W$ from f to g, q.e.d.

Proposition 2. Let R, V, Ω and κ be as in Theorem 3 (a). Let f be a quasiconformal mapping of R whose Beltrami coefficient is equal to $\zeta\kappa$ on Ω for some $\zeta \in \Delta$. Then, for every quasiconformal mapping g of R

onto $f(R)$ with $g \simeq f : R \to f(R)$, its Beltrami coefficient μ_g satisfies

$$\|\mu_g | R-V\| \geqq |\zeta|.$$

Proof. The proof is divided into three steps.

1) Take $r_1 \in (0,1) - \{|a_n| ; n \in \mathbb{N}\}$ so that $\mathrm{Cl}(g(p^{-1}(\Delta'_{r_1}))) \subset f(\Omega)$, where $\mathrm{Cl}(\cdot)$ denotes the closure. Set $\Omega_1 = p^{-1}(\Delta'_{r_1})$ and let R_1 be an arbitrary component of $\Omega - \mathrm{Cl}(\Omega_1)$. Then R_1 is topologically finite, and the border $b(R_1)$ is divided into two parts $b = b(R_1) \cap b(\Omega)$ and $b_1 = b(R_1) \cap p^{-1}(\{|z| = r_1\})$. We first claim that there is a quasiconformal mapping g_1 of R_1 onto a component R_1' of $f(\Omega) - \mathrm{Cl}(g(\Omega_1))$ such that $g_1 \simeq f|R_1 : R_1 \to f(\Omega)$, $g_1 = f$ on b and $g_1 = g$ on b_1. In fact, since R_1 and R_1' are of the same type, there is a quasiconformal mapping g_1 of R_1 onto R_1' with $g_1 \simeq f|R_1$ (cf. Fehlmann [5]). Furthermore, by Lehto-Virtanen [18, p.96] or Kelingos [12, Theorem 1], such g_1 can be deformed in an annular half-neighborhood of each component of $b(R_1)$ so that $g_1 = f$ on b and $g_1 = g$ on b_1.

2) The above g_1's for all the components of $\Omega - \mathrm{Cl}(\Omega_1)$ and

$g|Cl(\Omega_1)$ define a quasiconformal mapping g_2 of Ω onto $f(\Omega)$ such that $g_2|\Omega_1 = g|\Omega_1 \simeq f|\Omega_1$, $g_2|R_1 = g_1|R_1 \simeq f|R_1$ for each component R_1 of $\Omega - Cl(\Omega_1)$ and $g_2 = f$ on $b(\Omega)$. Note that, by Lemma 7, as a homotopy from $f|\Omega_1$ to $g|\Omega_1$, we can take one whose range is in $f(\Omega)$. By Lemmas 5 and 6 we obtain a quasiconformal mapping g_3 of Ω onto $f(\Omega)$ such that $g_3 \simeq f : \Omega \to f(\Omega)$ rel $b(\Omega)$ and $g_3 = g$ on $\Omega_2 = p^{-1}(\Delta'_{r_2})$ for some r_2, $0 < r_2 < r_1$.

3) Let ν, ν_1 be the Beltrami coefficients of g_3, g_3^{-1}, respectively, and set $k = \|\nu|\Omega_2 - V\|$. We have $k = \|\mu_g|\Omega_2 - V\| \leqq \|\mu_g|R - V\|$, and $k = \|\nu_1 \circ g_3|\Omega_2 - V\|$ Let $\{\phi^*_n\}_{n=1}^{\infty}$ be a sequence of quadratic differentials on Ω obtained by lifting $\{\phi(\cdot, a_n)\}_{n=1}^{\infty}$, where $\{a_n\}$ is the sequence in the definition (4.2) of Ω and $\phi(\cdot,\cdot)$ is the integrable holomorphic quadratic differential on $\Delta' - \{a_n ; n \in \mathbb{N}\}$ defined in Lemma 3. Then, since all points of $p^{-1}(a_n)$ are branch points (or punctures) and m = the number of sheets of the covering p is finite, every ϕ^*_n is holomorphic and integrable, in fact, $\|\phi^*_n\|_1 = m\|\phi(\cdot, a_n)\|_1 = 2m\pi|\log|a_n|| + O(1)$, by Lemma 3. To show Proposition 2, we may assume $\zeta \neq 0$, for otherwise the assertion is trivial.

Set $\phi_n = (|\zeta|/\zeta)\phi_n^*/\|\phi_n^*\|_1$ Obviously ϕ_n converges to zero uniformly on $\Omega - \Omega_2$, and by Lemma 3, we have

$$\iint_{V\cap\Omega} |\phi_n|\,dxdy \leqq \|\phi(\cdot,a_n)\|_1^{-1}\iint_{p(V\cap\Omega)} |\phi(z,a_n)|\,dxdy$$

$$= o(1)$$

and

$$\iint_{\Omega} \zeta\kappa\phi_n\,dxdy = \frac{|\zeta|}{\|\phi(\cdot,a_n)\|_1}\iint_{\Delta} \frac{z}{\bar{z}}\phi(z,a_n)\,dxdy$$

$$= |\zeta| + o(1)$$

Hence $\{\phi_n\}$ is a Hamilton sequence for $\zeta\kappa|\Omega$. We have by Theorem E

$$1 \leqq \frac{1+k}{1-k}\,\frac{1}{1-|\zeta|^2}\iint_{\Omega_2-V} |\phi_n|\left|1-\zeta\kappa\frac{\phi_n}{|\phi_n|}\right|^2 dxdy$$
$$+ \frac{1+k'}{1-k'}\,\frac{1+|\zeta|}{1-|\zeta|}\iint_{(\Omega-\Omega_2)\cup(\Omega\cap V)} |\phi_n|\,dxdy.$$

where $k' = \|\nu_1\|$. We see by the same way as in the proof of Theorem 2 that the first integral is equal to $(1-|\zeta|)^2 + o(1)$ and the second $o(1)$.

Thus we obtain

$$1 \leqq \frac{1+k}{1-k}\,\frac{1-|\zeta|}{1+|\zeta|}, \qquad \text{or} \qquad k \geqq |\zeta|.$$

This completes the proof.

Proof of Theorem 3. (a): Suppose that $\nu \sim \zeta\kappa$ for some $\nu \in M_1(V,R)$

and some $\zeta \in \Delta$. Then by Proposition 2 above we have $|\zeta| \leqq \|\nu|R-V\| = 0$, thus $\Phi(M_1(V,R)) \cap \Phi(\Delta(\kappa)) = \{0\}$. Proposition 2 also yields that $\zeta\kappa$ is extremal in the class $\{ \nu \in M_1(R) \ ; \ \nu \sim \zeta\kappa \}$. Hence $d_\Delta(0,\zeta) = d_T(0,\Phi(\zeta\kappa))$ $(= d_T([0],[\zeta\kappa]))$.

For $\zeta \in \Delta$ we denote by f_ζ a quasiconformal mapping of R whose Beltrami coefficient is $\zeta\kappa$. The quasiconformal mapping f_ζ is conformal in $R - \mathrm{Cl}(\Omega)$ and $f_\zeta|\Omega$ is projected to a quasiconformal mapping F_ζ of Δ' whose Beltrami coefficient is $\zeta z/\bar{z}$. We may assume that F_ζ is a self-mapping of Δ' with $F_\zeta(1) = 1$. Then the explicit form of F_ζ is

$$w = F_\zeta(z) = z \exp\left(\frac{2\zeta}{1-\zeta}\log|z|\right),$$

in particular, $zw_z = w/(1-\zeta)$. Let ζ and ζ' be in Δ, then the Beltrami coefficient of $F_{\zeta'} \circ F_\zeta^{-1}$ at $w = F_\zeta(z)$ is

$$\frac{\zeta' - \zeta}{1 - \bar{\zeta}'\zeta} \cdot \frac{z}{\bar{z}} \cdot \frac{w_z}{\overline{w_{\bar{z}}}} = \frac{1-\bar{\zeta}}{1-\zeta} \cdot \frac{\zeta' - \zeta}{1 - \bar{\zeta}'\zeta} \cdot \frac{w}{\bar{w}}$$

Consequently, by the same argument as above, we see that $f_{\zeta'} \circ f_\zeta^{-1}$ is extremal and $d_\Delta(\zeta,\zeta') = d_T(\Phi(\zeta\kappa),\Phi(\zeta'\kappa))$. Thus we have (a)

The proof of (b) is now easy. Noting that $t\kappa \in M_1(R^*)$ for $t \in I$

and that $T(\Gamma,\sigma)$ is isometrically embedded in $T(G,\hat{\mathbb{R}})$, where G is a Fuchsian model of Γ (cf. Earle [3]), we obtain (b) by (a), q.e.d.

Proof of Theorem 3' First, let us consider the case where $R = R^*$. Let $(\zeta_j) \in \Delta^N$ and $\nu \in M_1(R)$. If $\kappa = \sum_j \zeta_j \kappa_j \sim \nu$, then $\sup_j |\zeta_j| \leqq \|\nu|R-V\|$ by Proposition 2, in particular, κ is extremal. If furthermore $\nu \in M_1(V,R)$. then all ζ_j are zeros, hence we have (4.5). The proof for the case where $R \neq R^*$ is the same. The second conclusion follows from the same argument as in the proof of Theorem 3, q.e.d.

§6. Examples

We first give an example showing that the hypothesis of Corollary 1 is not quasiconformally invariant in the following sense:

Proposition 3 Let (Γ,σ) be an arbitrary pair of a Fuchsian group Γ and a boundary condition σ for which $\mathrm{Area}(D/\Gamma) = \infty$ and ω is bounded. Then there is a measurable subset E, invariant under Γ, of D with $\mathrm{Area}(E/\Gamma) = \infty$, and for each $K > 1$ there is a K-quasiconformal self-mapping f of $\hat{\mathbb{C}}$ such that the Beltrami coefficient of f belongs to $M_0(D-E,\Gamma,\sigma)$ and $\mathrm{Area}(f(E)/\Gamma) < \infty$.

Proof. In case of $\sigma \neq \hat{\mathbb{R}}$ (resp. $\sigma = \hat{\mathbb{R}}$), let P be a Dirichlet fundamental region whose center is in $\hat{\mathbb{R}} - \sigma$ (resp. in U and fixed by no elements in Γ). Since $\mathrm{Area}(P) = \infty$, there is a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of mutually disjoint hyperbolic disks with (hyperbolic) radii r_n and centers c_n such that $\Delta_n \subset P \cap U$,

$$(6.1) \qquad \sup_n r_n < \infty \quad \text{and} \quad \mathrm{Area}\Big(\bigcup_{n=1}^{\infty} \Delta_n\Big) = \pi \sum_{n=1}^{\infty} \sinh^2 r_n = \infty .$$

For each n, let π_n be a universal covering map : $\Delta = \{ |z| < 1 \} \to D$

with $\pi_n(0) = c_n$, and $\Delta'_n = \{ |z| < a_n \}$ be the component of $\pi_n^{-1}(\Delta_n)$ containing the origin. Then there is a sequence $\{b_n\}_{n=1}^{\infty}$ with $0 < b_n \leqq 1$ suth that $\sum_n (a_n b_n)^2 = \infty$ and $\sum_n (a_n b_n^K)^2 < \infty$ for all $K > 1$. In fact, from $a_n = \tanh r_n$ and (6.1) it follows that $a = \sup_n a_n < \infty$ and $\sum_n a_n^2 = \infty$, hence there is a sequence $\{n(j)\}_{j=1}^{\infty}$ of natural numbers such that $n(1) = 1$ and $a^2 \leqq \sum_{n=n(j)}^{n(j+1)-1} a_n^2 < 2a^2$ Let $b_n = j^{-1/2}$ for $j \geqq 1$ and $n(j) \leqq n < n(j+1)$, then $\{b_n\}$ is the required sequence.

Set $E_n = \pi_n(\{ |z| < a_n b_n \})$ and $E = \bigcup_{\gamma \in \Gamma} \gamma(\bigcup_n (E_n \cup \{z \in D \, ; \, \bar{z} \in E_n\}))$. Let g_n be a self-mapping of Δ'_n defined by $g_n(z) = b_n^{K-1} z$ for $|z| < a_n b_n$ and $g_n(z) = a_n^{1-K} |z|^{K-1} z$ for $a_n b_n \leqq |z| < a_n$, then g_n is K-quasiconformal. These mappings g_n's and the covering maps π_n's canonically induce a K-quasiconformal self-mapping f of $\bigcup_n \Delta_n$. Extend f to $\mathrm{Cl}(P \cap U)$ so that f fixes all points in $\mathrm{Cl}(P \cap U) - \bigcup_n \Delta_n$, and after, to P symmetrically (when $\sigma \neq \hat{\mathbb{R}}$), finally, to $\hat{\mathbb{C}}$ so that f is compatible with Γ The extended mapping f is well-defined and a K-quasiconformal self-mapping of $\hat{\mathbb{C}}$ whose complex dilatation belongs to

$M_0(D-E,\Gamma,\sigma)$ by definition. In addition,

$$\mathrm{Area}(E/\Gamma) \geqq \sum_n \mathrm{Area}(\{\,|z| < a_n b_n\,\})$$

$$\geqq \sum_n (a_n b_n)^2 = \infty ,$$

on the other hand,

$$\mathrm{Area}(f(E)/\Gamma) \leqq 2\sum_n \mathrm{Area}(g_n(\{\,|z| < a_n b_n\,\}))$$

$$= 2\sum_n \mathrm{Area}(\{\,|z| < a_n b_n^K\,\})$$

$$\leqq \mathrm{const.}\sum_n (a_n b_n^K)^2 < \infty.$$

Thus we have our assertion.

Our second example is concerned with Theorems 2 and 3.

Proposition 4. There exist a Riemann surface R which contains Ω satisfying (4.2), and a measurable subset V for which the condition (4.3) in Theorem 3 holds but the hypothesis (4.1) of Theorem 2 does not.

Proof. Consider the case Γ is the trivial group $\{id\}$ and $\sigma = \{\,2^n\,;\, n \in \mathbb{Z}\,\} \cup \{0, \infty\}$. Let $R = \hat{\mathbb{C}} - \sigma$ ($= D$) and $\Omega = \{\, z \in R\,;\, |z| < \sqrt{2}\,\}$. (One may consider the case where Γ is a Fuchsian group of the first kind such that $U/\Gamma = \mathbb{C} - (\{\,2^n\,;\, n \in \mathbb{Z}\,\} \cup \{\,0\,\})$. In such a case it is not necessary that V

is assumed to be symmetric with respect to $\mathbb{R}$.) Fix a δ, $0 < \delta < \pi$, and set $S = \{ z = re^{i\theta} \in R \ ; \ \delta \leqq \theta \leqq 2\pi - \delta \}$. We then claim that

$$(6.2) \qquad m = \inf\{ |z|^2 \lambda_R(z)^2 \omega(z) \ ; \ z \in S \} > 0.$$

Let T be the conformal self-mapping of $R : z \mapsto 2z$, and $X = S \cap \{1/2 \leqq |z| \leqq 1\}$. Since $\omega(z)$ as well as $|z|^2\lambda_R(z)^2$ is invariant under $\langle T \rangle$ by Theorem B and the definition of $\omega(z)$, we have $m = \inf \{ |z|^2\lambda_R(z)^2\omega(z) \ ; \ z \in X \}$. The set X is compact in R, the function ω is lower semi-continuous and $\inf_X \lambda_R^2 > 0$, hence if $m = 0$ then there is a point z_0 in X at which ω vanishes. This implies that $F(z_0, \zeta) = 0$ for all ζ in R. It follows from the reproducing property of F in Theorem B that all functions in $A^1(R, \{id\})$ vanish at z_0, but this is absurd because $1/\{z(z-1)(z-2)\}$ belongs to $A^1(R, \{id\})$ Thus we see (6.2).

Let $\theta : [0, \sqrt{2}] \to [0, \pi - \delta]$ be a continuous function such that $\theta(0) = 0$ and $\int_{(0,\sqrt{2}]} (\theta(t)/t)\,dt = \infty$. Let V be a measurable subset of $S \cap \Omega$ such that V is symmetric with respect to $\mathbb{R}$ and $\int_{V \cap \{|z|=r\}} d\theta = 2\theta(r)$. These R and V are what we seek, in fact, we have

$$\iint_{p(V\cap\Omega)-\Delta_r} |z|^{-2}dxdy = 2\int_{\sqrt{2}r}^{\sqrt{2}} t^{-1}\theta(t)dt = o(|\log r|) ,$$

where $p(z) = z/\sqrt{2}$, and

$$\int_V \omega dA \geqq m\iint_V |z|^{-2}dxdy = \infty .$$

This completes the proof.

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